Advanced ODE-Lecture 11
Dynamic Systems

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Motivation

- What is the characterization of a dynamic system: algebraic or geometric structure? It is very important!
- Equilibrium, closed trajectory and open trajectory are three important classes of trajectories.
- Classification of trajectories of dynamic systems indicates the research directions of dynamic systems.

Dynamic Systems

Consider autonomous systems

$$\dot{x} = f(x), \tag{11.1}$$

where $f: D \to R^n$ is locally Lip., $D \subseteq R^n$ and $f(0) = 0$. Assume that (11.1) is complete for each $x_0 \in R^n$, i.e. it is no blow-up for each $x_0 \in R^n$. Such a complete autonomous system is called **dynamic systems**.

Remark 11.1 Can we treat the solution set of dynamic systems (11.1) as the one of linear systems to find some algebraic properties of the solution set? We hope so, but it is difficult. We can only find some group property. In fact, we cannot expect all because the dynamic systems (11.1) is much complicated than linear systems.

Definition 11.1 Let $x(t; x_0)$ be represented as a solution of (11.1), passing through the point $(0, x_0)$. $\{x(t; x_0) \mid t \in R\} \subseteq R^n$ is called a **trajectory** or **orbit** of (11.1), which is denoted as $\gamma(x_0)$. We denote $\gamma^+(x_0)$ **as a positive orbit** defined by $x(t; x_0)$, $t \geq 0$; $\gamma^-(x_0)$ **as a negative orbit** defined by $x(t; x_0)$, $t \leq 0$. Then, $\gamma(x_0) = \gamma^+(x_0) \cup \gamma^-(x_0)$. If $x(t; x_0)$ is periodic, $\gamma^+(x_0) = \gamma^-(x_0)$.

Remark 11.2 Look at the difference between a solution $x(t; x_0) \in R \times R^n$ of (11.1) and a trajectory $x(t; x_0) \in R^n$ of (11.1), where t is regarded as a parameter, although their notations are the same.

Three Fundamental Properties

Superposition Principle is a characteristic of linear systems. What characteristics of dynamic systems are? It has three properties.

Lemma 11.1 (Translation Property) If $x(t; t_0, x_0)$ is a solution of (11.1), then so is $x(t+s; t_0, x_0)$ for any constant $s \in R$.

Proof. Since

$$x'(t+s; t_0, x_0) = \frac{dx(t+s; t_0, x_0)}{dt} = \frac{dx(t+s; t_0, x_0)}{d(t+s)} = f(x(t+s; t_0, x_0)),$$

notice that $I_{\max} = R$ for all $x_0 \in D$, so $t+s \in R$. Therefore, $x(t+s; t_0, x_0)$ is also a solution of (11.1). □

Lemma 11.2 (Uniqueness of Trajectory) If two trajectories of (11.1) intersect at a point, then, the two trajectories are identical.

Proof. Suppose that two trajectories $x(t; t_0, x_0)$ and $\tilde{x}(t; t_1, x_0)$ intersect at a point $x_0 \in R^n$, it must be $t_1 \neq t_0$. Otherwise, $x(t; t_0, x_0) \equiv \tilde{x}(t; t_1, x_0)$ by uniqueness of solution.

By Translation Property, it follows that $x(t + t_0 - t_1; t_0, x_0)$ is also a solution of (11.1). Moreover, it has

$$x(t + t_0 - t_1; t_0, x_0)|_{t=t_1} = x_0 = \tilde{x}(t; t_1, x_0)|_{t=t_1}.$$

By uniqueness of solution, it follows that $x(t + t_0 - t_1; t_0, x_0) \equiv \tilde{x}(t; t_1, x_0)$. Therefore, they are identical in R^n. The uniqueness of trajectory is proved. □

Remark 11.3 Lemma 11.1 and Lemma 11.2 show that two solutions $x(t; t_0, x_0)$ and $x(t+s; t_0, x_0)$ may have different time to pass through x_0, but they have the same trajectory.

Remark 11.4 Since (11.1) is time-invariant, $x(t-t_0; 0, x_0)$ is also a solution of (11.1) if $x(t; t_0, x_0)$ is a solution of (11.1). Moreover,

$$x(t-t_0; 0, x_0)|_{t=t_0} = x_0 = x(t; t_0, x_0)|_{t=t_0}.$$

Then, $x(t; t_0, x_0) \equiv x(t-t_0; 0, x_0)$ by uniqueness of solution. Therefore, taking $t_0 = 0$, without loss of generality, we denote $x(t; t_0, x_0)$ as $x(t; x_0)$. The trajectory of (11.1) is determined uniquely by its initial state x_0, independent of t_0.

Lemma 11.3 (Group Property) $x(t_1+t_2;\, x_0) = x(t_2;\, x(t_1;\, x_0))$, for $x_0 \in R^n$.

Proof. Since $x(t+t_1;\, x_0)$ and $x(t;\, x(t_1;\, x_0))$ are both solutions of (11.1), satisfying

$$x(t;\, x(t_1;\, x_0))\,|_{t=0} = x(t_1;\, x_0) = x(t+t_1;\, x_0)\,|_{t=0}\,,$$

they are identically equal by uniqueness of solution. Taking $t = t_2$ both for $x(t+t_1;\, x_0)$ and $x(t;\, x(t_1;\, x_0))$ results in the desired result. □

Remark 11.5 Denote $x_1 = x(t_1;\, x_0)$ and $x_2 = x(t_1+t_2;\, x_0)$. Lemma 11.3 shows that a trajectory has an additive property for time t. O a trajectory satisfies a state transition property.

Remark 11.6 Lemma 11.1-11.3 give a basic characterization of a dynamic system, which is not true for time-varying systems in general. So a time-varying system is not a dynamic system. Otherwise, we have to change the definition of a dynamic system.

Remark 11.7 Why is Lemma 11.3 called “Group Property”?

Define a transformation $x_t : x_0 \in D \to x(t; x_0) \in D$, where $t \in R$ is a parameter.

Then, we have a transformation set as follows.

$$T = \{x_t \mid x_0 \to x(t; x_0),\ t \in R\}.$$

We define an (abstract) addition of transformations as follows.

$$x_{t_1} + x_{t_2} =: x(t_2; x(t_1; x_0)) \quad \text{for any} \quad x_{t_1}, x_{t_2} \in T.$$

This defined addition is well defined because of Lemma11.3, satisfying:

- Associate law: $(x_{t_1} + x_{t_2}) + x_{t_3} = x_{t_1} + (x_{t_2} + x_{t_3})$;
- Zero Element: $x_{t=0} \in T$: $x_{t=0} = x(0; x_0) = x_0$;
- Inverse Element: x_{-t}: $x_{-t} = x(-t; x_0)$, for any $x_t = x(t; x_0) \in T$ such that

$$x_t + x_{-t} = x(t; x(-t; x_0)) = x(t + (-t); x_0) = x(0; x_0) = x_0 = x_{t=0}.$$

Therefore, $T = \{x_t \mid x_0 \to x(t; x_0),\ t \in R\}$ is a group on the defined addition, which is said **a continuous transformation group on a single parameter** t. Since Lemma 13.3 plays a key role for such a group, it is also said “a group property”.

Classification of Trajectories of Dynamic Systems

1) Periodic Solution and Limit Cycle

If a solution $x(t)$, $t \in R$, of the system (11.1) satisfies $x(t+\omega) = x(t)$, $\forall t \in R$, $\omega \neq 0$, then, $x(t)$ is a **periodic solution** of the system (11.1), where ω is a period.

The trajectory of a periodic solution $x(t)$ is called a **cycle** of (11.1).

Remark 11.8 Since constant solution of (11.1) is also periodic, it is trivial because it has no smallest positive period. Its corresponding trajectory is equilibrium. So the trajectory of a non-trivial periodic solution is called a **closed trajectory** or a **limit cycle.**

Lemma 11.4 any non-trivial periodic solution $x(t)$ associated to a closed trajectory of the dynamic system (11.1) must have a smallest positive period.

Proof. Let T be a set of all positive periods of a periodic solution of (11.1). Denote $\omega_0 = \inf\{\mathrm{T}\}$ and then, $\omega_0 \geq 0$. We want to show $\omega_0 > 0$. If $\omega_0 = 0$ by contradiction, then there exists a sequence $\{\omega_k\} \subset \mathrm{T}$ such that $\lim_{k\to\infty} \omega_k = \omega_0 = 0$, as well as $x(t+\omega_k) = x(t)$, $\forall\, t \in R$. Then

$$f(x(t)) = x'(t) = \lim_{k\to\infty} \frac{x(t+\omega_k) - x(t)}{\omega_k} = 0, \quad \forall t \in R,$$

which implies that $x(t)$ is a constant solution. It is trivial. This is a contradiction to non-trivial assumption. Next to show $\omega_0 \in \mathrm{T}$. That is to show that T is a closed set.

Since $x(t)$ is continuous, we take limit on both sides of $x(t+\omega_k) = x(t)$ to get

$$x(t+\omega_0) = x(t), \quad \forall t \in R.$$

This shows that $\omega_0 \in \mathrm{T}$. □

2) Classification of Trajectories

Theorem 11.5 Suppose that ξ is not equilibrium ($f(\xi) \neq 0$). Then

1) $x = x(t;\xi)$ is a closed trajectory; or

2) $x = x(t;\xi)$ is a trajectory that does not intersect itself. This type of trajectory is also called as an open trajectory.

Proof. 1) If there exist t_1, t_2 with $t_1 < t_2$ such that $x(t_1;\xi) = x(t_2;\xi)$, then, $x = x(t;\xi)$ is not a constant solution of (11.1). Otherwise, its corresponding closed trajectory is only one point ξ, which means that it is equilibrium. This is a contradiction of the assumption. So we can assume that

$$\forall s \in (t_1, t_2), \quad x(s;\xi) \neq x(t_1;\xi) = x(t_2;\xi).$$

Now we consider two solutions as follows.

$$x(t+t_1;\xi)\,;\quad x(t+t_2;\xi),\ \ \forall t\in R.$$

Since

$$x(t+t_1;\xi)\,|_{t=0}=x(t_1;\xi)=x(t_2;\xi)=x(t+t_2;\xi)\,|_{t=0}\,,$$

By uniqueness,

$$x(t+t_1;\xi)\equiv x(t+t_2;\xi),\ \ \forall t\in R.$$

It results in

$$x(t;\xi)\equiv x(t+\omega;\xi)\,,$$

where $\omega=t_2-t_1>0$, $\forall t\in R$. So $x=x(t;\xi)$ is a non-trivial periodic solution with a period $\omega=t_2-t_1>0$. It associates a closed trajectory.

1) If for any $t_1\neq t_2$, it has $x(t_1;\xi)\neq x(t_2;\xi)$. Then the following transformation:

$$x(\cdot\,;\xi):\ t\in R\rightarrow x(t;\xi)\in l$$

is a continuous map. Its corresponding trajectory $l=x(R;\xi)$ is a trajectory that does not intersect itself for all $t\in R$. □

Remark 11.9 This theorem shows that the trajectory types in the trajectory set of the dynamic system (11.1) can be classified by 3 different types of trajectories: equilibrium; closed orbit (trajectory) and open orbit (trajectory). They are concerned by each for the relationship with their nearby trajectories.

Remark 11.10 For equilibrium, it refers to Lyapunov stability method. For closed orbit, it refers to limit cycle theory and the theory of periodic solutions. For open orbit, it refers to the study an asymptotic behavior of a trajectory, which determines the form of an open orbit and where to go, which caused the introduction of limit point notions.

We also concern overall relation among trajectories from equilibrium, closed orbits and open orbits. Therefore, it is possible to make a distribution about all trajectories of a dynamic system.

Summary

1) The characteristic of a dynamic system is 3 fundamental properties.

2) The whole trajectories of a dynamic system can be classified as 3 types of trajectory: equilibrium, closed and open trajectory.

3) The research directions are now clear under the classification. Local dynamic behaviors of each type of trajectory should be studied first. Then to study whole set of trajectories based on local dynamic behaviors. This content is referred to the geometry theory of ODE, or quantitive analysis of dynamic systems, in which Lyapunov stability theory has its own characteristics, which develops very well in control sciences and the other subjects.

Homework

- Review today's contents of class.